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SOME AXIOMS AND THEOREMS IN DAMAGE MECHANICS AND FATIGUE OF MATERIALS

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Abstract—The treatments of problems in continuum damage mechanics (CDM) that appear in the scientific literature are based on a number of more or less explicit assumptions. In this paper, these assumptions are reformulated and shown to form a single axiomatic framework of both the "internal variables" and the "multifield continuum" schemes: the fundamental features of both descriptions are thus recognized and correlated with the general principles of mechanics. These developments are based on the general theory of actions developed by Coleman and Owen (Coleman, B. D. and Owen, D. R. (1974). A mathematical foundation for thermodynamics. *Archives of Rational Mechanical Analysis* **54**, 1–104; (1975). On thermodynamics and elastic plastic materials. *Archives of Rational Mechanical Analysis* **59**, 25–51; (1977). On thermodynamics of semi-systems with restrictions on the accessibility of states. *Archives of Rational Mechanical Analysis* **66**, 173–181), which is presented in Section 2 in a way slightly different from usual. Then, five axioms for CDM are introduced in Section 3, and related theorems and corollaries are proved. Finally, the Palmgren–Miner's rule of fatigue damage and its limitations are reformulated in the context of the established general framework. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Many usual models of continuum damage mechanics (CDM) are based on the internal variable scheme (e.g., Bazant, 1986; Bazant and Pijauder-Cabot, 1988; Chen and Schreyer, 1994; Francfort and Marigo, 1993; Krajcinovic, 1985; 1989; Krajcinovic and Fonseka, 1981; Krajcinovic *et al.*, 1993; Lemaitre, 1994; Lubarda and Krajcinovic, 1995; Pijauder-Cabot and Banallal, 1993; Simo and Ju, 1987; Smyshlyaev and Willis, 1996). Such models require: (a) the preventive choice of damage variables (Budianski and O'Connell, 1976; Onat and Leckie, 1988; Lubarda and Krajcinovic, 1993; Woo and Li, 1994), (b) the choice of an expression of the free energy compatible with the rules of thermodynamics and with the axiom of frame indifference; (c) the introduction of kinetic equations which govern the rate of the damage variables. In these works, the attention has been focused either on the microscale (the microscopic structure of the body and the fluctuations around the inhomogeneities of the stress and strain fields have been considered; e.g., Altus, 1991; Krajcinovic *et al.* 1991; Kunin, 1993; Nemat-Nasser and Hori, 1993; Nemat-Nasser *et al.* 1993; Ostoja-Starzewski, 1993), on the macroscale or on the mesoscale. The existence of a potential in the damage phenomenon has been considered acceptable for physical reasons (Krajcinovic, 1989). A potential damage surface, in analogy with the plasticity theory, has been used to establish criteria of damage growth (e.g., Lubarda and Krajcinovic, 1995).

A different point of view on the damage has been developed recently by using the multifield continuum description (Augusti and Mariano, 1995, 1996; Mariano, 1995, 1996a). In this approach the microcrack state is described by a field that satisfies appropriate balance equations. The introduction of an additional entropy flux in the Clausius–Duhem inequality allows to establish consistent criteria of damage growth without using potential damage surfaces. From a slightly different point of view, an analogous direction has been followed by Markov (1995) and Frémond and Nedjar (1996): the numerical simulations of the latter authors are a significant addition to the theoretical speculations about multifield description.

The aim of the present paper is to formulate in an explicit way the basic assumptions under the models of either type, and moreover to show a single overall framework, in which the fundamental features of the damage—independent of the specific rheological model assumed—can be easily recognized, together with their relation with the general principia of mechanics.

The mathematics used in these developments are based on the general thermodynamical theory of actions developed by Coleman and Owen (1974, 1975, 1977), which is summarized—in a way slightly different from usual—in Section 2. Mathematical objects are introduced which have the structure of the typical instruments of continuum mechanics (work, energy, etc) and are independent of a specific expression related to a rheological or a mechanical (internal variables, microstructured continua, ...) model. For example, in order to fix the dissipative nature of the internal work it suffices to use an object that has the general properties of the work without assuming a specific form for it.

It is thus possible to formulate (in Section 3) five general axioms for the whole CDM, and prove a number of other propositions. The Palmgren–Miner's rule for fatigue is used in Section 4 as an illustration of the possibilities of the proposed formulation.

2. BASIC ASSUMPTIONS AND DEFINITIONS

In this section, relevant concepts and definitions formulated by Coleman and Owen (1975, 1977), necessary for further developments, are explicitly presented, together with the formalism used in the present paper.

2.1. *States and processes*

The finite set of independent fields which describe a continuous body, $\mathscr{B}$, at a given instant t is called its *state* and indicated by σ. For instance, in an internal variable model for brittle thermoelastic solids with isotropic damage, σ is the three-plet of fields

$$\sigma = (\mathbf{F}, \theta, D) \tag{1}$$

in which $\mathbf{F}$, θ and D are, respectively, the field of the deformation gradient, of the absolute temperature, of the scalar damage; in models with microstructure for ductile solids with microcracks, σ can be defined as

$$\sigma = (\mathbf{E}, \mathbf{E}^p, \theta, \mathscr{N}) \tag{2}$$

in which $\mathbf{E}$ is the field of the deformation tensor, $\mathbf{E}^p$ the field of the plastic deformation tensor, θ has been already defined, $\mathscr{N}$ the dipole approximation of the microcrack density function (Lubarda and Krajcinovic, 1993; Augusti and Mariano, 1995).

Note that, differently from Noll (1972), σ is defined as the state of the whole body, not the state of a point of the body. It is a point of a functional space Σ, called the state space, which by assumption has at least the trivial topological properties necessary for simple operations to make sense.

The interaction of the body, $\mathscr{B}$, with the external environment is represented by a process $P = P^t$, $t \in [0, d_p]$, i.e., an operator that acts on Σ during a finite time interval $[0, d_p]$. For instance, P can be a deformation, a load, a thermal history etc. With reference to internal variable models, P accounts only for the modifications of observable variables due to interference with the external environment. P cannot represent, for example, an internal variable history. The set of all possible P is indicated by Π.

Every process P maps Σ into itself:

$$P: D(P) \to R(P); \quad D(P) \subseteq \Sigma; \quad R(P) \subseteq \Sigma.\dagger \tag{3a,b,c}$$

A generic process P induces in Σ a state transformation ρ_P, i.e., a change from the

† $D(P)$ and $R(P)$ are the domain and the range of P, respectively.

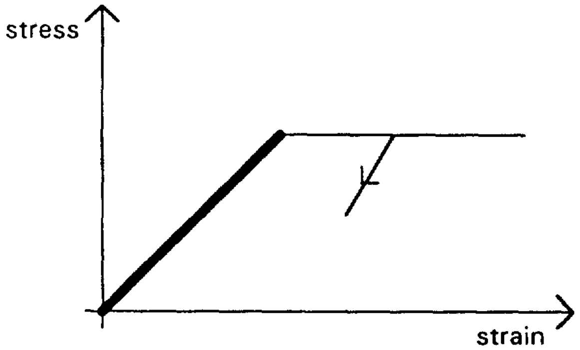

Fig. 1. Monoaxial stress–strain relationship for elastic-perfectly plastic materials: the bold line is the set of all base states.

initial state σ to a final state $\rho_P\sigma$ at $t = d_P$. $\rho_{P'}\sigma$ represents the path induced in Σ by P from σ, i.e., the set of all states encountered during the state transformation. It represents the thermodynamics evolution of the body $\mathscr{B}$, subjected to P.

For the pair (Π, Σ) two fundamental properties are required.

(I) Approachability: there exists at least a state $\bar{\sigma}$ such that it is possible to approach† from it any state in Σ. In symbols: there exists $\bar{\sigma}$ such that the set

$$\Pi\bar{\sigma} \overset{\text{def}}{=} \{\rho_P\bar{\sigma} \mid P \in \Pi, \bar{\sigma} \in D(P)\} \tag{4}$$

is dense in Σ.

Each state like $\bar{\sigma}$ is called a base state for the space Σ (a very simple example is shown in Fig. 1).

(II) Composition of the processes (e.g., superposition of loads): in the set

$$\mathscr{P} \overset{\text{def}}{=} \{(P'', P') \in \Pi \times \Pi \mid D(P'') \bigcap R(P') \neq \varnothing\} \tag{5}$$

there exists a map from $\Pi \times \Pi$ into Π, $(P'', P') \to P''P'$, such that

$$D(P''P') = \rho_{P'}^{-1}(D(P'') \bigcap R(P')) \tag{6}$$

and $\forall \sigma \in D(P''P')$, $\rho_{P''P'}\sigma = \rho_{P''}\rho_{P'}\sigma$.

It is also understood that the processes are applied in succession to the body.

2.2. *Actions*

All functionals like power, work etc. which associate a real number to each path in the state space, can be considered as particular cases of the general concept of action. In fact, given the set of compatible pairs (P, σ), indicated by $\Pi \diamondsuit \Sigma$ and defined by

$$\Pi \diamondsuit \Sigma \overset{\text{def}}{=} \{(P, \sigma) \in \Pi \times \Sigma \mid \sigma \in D(P)\} \tag{7}$$

a real functional $a(\cdot, \cdot) : \Pi \diamondsuit \Sigma \to \mathscr{R}$ is called action if it satisfies the following properties:

(a) Additivity with respect to the composition of processes, i.e.,

$$a(P''P', \sigma) = a(P', \sigma) + a(P'', \rho_{P'}\sigma) \tag{8}$$

with $(P'', P') \in \mathscr{P}$ and $\sigma \in D(P''P')$.

(b) Continuity with respect to the states, i.e., $a(P, \cdot) : D(P) \to \mathscr{R}$ is continuous, for any $P \in \Pi$.

† For a rigorous mathematical definition of approachability see Coleman and Owen (1974, 1975, 1977).

For a body that can be described by the displacement field and its gradient only (simple body (Noll, 1972)), $a(\cdot,\cdot)$ can be

$$a(P,\sigma) = \int_0^{d_P}\int_{\mathscr{B}} \mathbf{T}_R : \dot{\mathbf{F}}\,\mathrm{d}V\mathrm{d}t + \int_0^{d_P}\int_{\mathscr{B}} \mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V\mathrm{d}t \tag{9}$$

in which $\mathbf{T}_R$ is the stress tensor, $\mathbf{T}_R : \dot{\mathbf{F}}$ the power density of internal forces and $\mathbf{b}\cdot\mathbf{v}$ is the power density of the body forces, $\mathbf{b}$.

For a damaged body, with reference to a model with internal variables and in the case of small deformations, a possible $a(\cdot,\cdot)$ can be expressed by

$$a(P,\sigma) = \int_0^{d_P}\int_{\mathscr{B}} \mathbf{T} : \dot{\mathbf{E}}\,\mathrm{d}V\mathrm{d}t + \int_0^{d_P}\int_{\mathscr{B}} \left(\mathbf{T}_y + \frac{3}{2X_\infty}\mathbf{X}^D : \mathbf{X}^D\right)\dot{p}\,\mathrm{d}V\mathrm{d}t \tag{10}$$

in which $\mathbf{T} : \dot{\mathbf{E}}$ is the power density of internal forces, $\mathbf{T}_y$ the yield stress, $\dot{p}$ the time rate of accumulated plastic strain, X_∞ the kinematic hardening material parameter, $\mathbf{X}^D$ the tensorial kinematic hardening stress variable (Lemaitre, 1992).

If an action is approximately positive when calculated along paths which are quasi-cycles starting from a given state σ or, more formally, if for any $\varepsilon > 0$ there is a neighborhood of σ, $I(\sigma)$, such that whenever $\rho_P\sigma \in I(\sigma)$, $a(P,\sigma) > -\varepsilon$, then $a(\cdot,\cdot)$ is said to have the dissipation property at the state σ.

An action $a(\cdot,\cdot)$ has the conservation property at the state σ if its value is zero on all cycles at σ.

2.3. *Potentials*

The internal energy is a potential for a specific action. In the same way the entropy is a lower potential for another specific action; i.e. the difference between the value of the entropy in the final state $\rho_P\sigma$ of a path and its value in the initial state σ is smaller than the action calculated along the same path.

Formally, a function $Z : \Sigma \to \mathscr{R}$ is a lower potential for an action $a(\cdot,\cdot)$ if

(1) the domain of Z is dense in Σ;

(2) $\forall\sigma_1$ and $\sigma_2 \in \Sigma$, and $\forall\varepsilon > 0$, an open neighborhood $I(\sigma_2)$ exists such that, for every P such that $\rho_P\sigma_1 \in I(\sigma_2)$ the inequality

$$Z(\sigma_1) - Z(\sigma_2) < a(P,\sigma_1) + \varepsilon \tag{11}$$

holds.

In the state space Σ, each constitutive equation is represented by a map on $\Sigma \times \Pi$ which takes values in an appropriate functional space. For example, in the Cauchy continuum the stress tensor $\mathbf{T}$ is represented by a map from $\Sigma \times \Pi$ into the space of the second order symmetric tensors.

3. AXIOMATIC FRAMEWORK FOR DAMAGED MATERIALS

By using the concepts and definitions of Section 2, five axioms can be formulated and related theorems demonstrated: these form a basic framework for damage theories, sufficient to justify some assumptions underlying usual damage models. The five axioms are elementary physical requirements which are, explicitly or implicitly, at the basis of all damage models.

However, it has not yet been demonstrated that they form a sufficient minimal set of axioms.

3.1. *Admissible states*

For a given body, let an admissible state be a state not violating an appropriate failure criterion.

The set of admissible states is indicated by $\mathscr{A}$.

> Axiom 1 (Closure of the admissible set of states): $\mathscr{A}$ is a non-empty proper subset of Σ, closed with respect to the topology in which the stress map† $\hat{\mathbf{T}}(P, \sigma)$ is uniformly continuous with respect to the states.

The boundary of $\mathscr{A}$ is constituted by the ultimate states considered for the body. For example, the boundary of $\mathscr{A}$ can represent the loss of integrity of the body. In this way the boundary of $\mathscr{A}$ represents an ultimate condition of failure, beyond which the analysis of the body in the form considered in $\mathscr{A}$ does not make sense. It is fundamental to note that admissible regions defined by damage potential surfaces (as in current technical literature) are proper subsets of $\mathscr{A}$.

> Axiom 2 (Axiom of integrity): The set of all base states of Σ is in $\mathscr{A}$. Such base states are attainable from each other in reversible manner.

Axiom 2 establishes the existence in $\mathscr{A}$ of at least one state which can undergo any type of damage. The existence of base states outside $\mathscr{A}$ is physically unacceptable with respect to the described phenomenon.

In the following, the symbol $\{a_\sigma \to \mathscr{J}\}$ represents the set of all real values of the action $a(\cdot, \sigma)$ calculated along all possible paths in $\mathscr{A}$ starting from $\sigma \in \mathscr{A}$ and going in $\mathscr{J}$, a proper subset of $\mathscr{A}$.

> Axiom 3 (Energy and dissipation to failure): The energy required to let the body fail (i.e., to bring it outside $\mathscr{A}$) is finite; in this state transformation the variation of entropy is also finite.
>
> Formally this Axiom can be expressed in the following way: two actions exist, $a'(\cdot,\cdot)$ conservative and $a''(\cdot,\cdot)$ dissipative, both at the base states (i.e., the first and the second law of thermodynamics); each of these actions is inf-bounded‡, i.e.:
>
> $$|\inf\{a'_\sigma \to \partial\mathscr{A}\}| < \infty \tag{12}$$
>
> and
>
> $$|\inf\{a''_\sigma \to \partial\mathscr{A}\}| < \infty. \tag{13}$$

The previous property is valid for all paths (the actions are considered classically as line integrals along state transformations) that connect pairs of states which can be approachable each other in $\mathscr{A}$.

3.2. *Damaging processes*

In the following, P^{-1} indicates the inverse process, e.g., the load process with reversed sign, the cooling process etc. (formally $P^{-1} = P^{(d_P - t)}, t \in [0, d_P]$); while P_{rel} indicates the relaxation process of the body, i.e., $\rho_{P_{rel}}\sigma = \lambda(\sigma)$, where $\lambda(\sigma)$ is a relaxed state with respect to the removing of the applied process P (see Def. 6.1 and Def. 7.1 by Coleman and Owen (1974) and Axiom V by Noll (1972)).

† $\hat{\mathbf{T}}$ associates to every state σ the stress tensor field $\mathbf{T}$ related to P. In the case of a microstructured continuum, the stress map associates to every state σ the pair $(\mathbf{T}, \mathscr{s})$ in which $\mathbf{T}$ is the macrostress and $\mathscr{s}$ the microstress acting on the microstructure (Capriz, 1989).

‡ Actually, it is only possible to assume the existence and inf-boundedness of $a'(\cdot,\cdot)$: in fact, it is very simple to demonstrate in such conditions the existence of natural entropies which can be defined without resorting to an assumed probability measure and calculated along paths representing state transformations of damage evolution (Mariano, 1966b).

Moreover, the set of cyclical processes is represented by

$$(\Pi \diamondsuit \Sigma)_{cycl} \stackrel{\text{def}}{=} \{(P, \sigma) \mid (P, \sigma) \in \Pi \diamondsuit \Sigma, \rho_P \sigma = \sigma\} \tag{14}$$

Definition 1. The process $P \in \Pi$, starting from a given state $\sigma_0 \in \mathring{\mathscr{A}}$ (where $\mathring{\mathscr{A}}$ is the interior of $\mathscr{A}$) is a damaging process with respect to the action $a(\cdot, \cdot)$, if

(a) the process and its reversal, after relaxation, are not cyclical, i.e.,

$$(P, \sigma_0), (P^{-1}P, \sigma_0), (P_{rel}P, \sigma_0) \notin (\Pi \diamondsuit \Sigma)_{cycl} \tag{15a,b,c}$$

(b) the minimum amount of energy required for the body failure, from the final state of the process, is smaller than from the initial state σ_0, i.e.,

$$\inf\{a_{\sigma_0} \to \partial \mathscr{A}\} \geqslant \inf\{a_{\sigma_2} \to \partial \mathscr{A}\} \tag{16}$$

where $\sigma_2 = \rho_{P^{-1}P}\sigma_0$, or $\sigma_2 = \rho_{P_{rel}P}\sigma_0$.

The equality sign holds when both "inf" are equal to zero.

In the following, P^0 indicates either P^{-1} or P_{rel}, and Π^0 indicates the set of damaging processes.

Axiom 4 (Possibility of damage): A body can be damaged starting from any state and it is not possible to return in $\mathscr{A}$ (restoration) once out.
Formally
(i) $\forall \sigma \in \mathscr{A}$, $\exists P \in \Pi$ such that $\sigma \in D(P)$ and $\rho_P \sigma \in \partial \mathscr{A}$; and
(ii) $\forall \sigma \in \mathscr{A}$, if $\rho_{P|_{t=\bar{t}}} \sigma \in \mathscr{A}$, then $\rho_{P|_{t=\bar{t}}} \sigma \in \mathscr{A}$, $\forall \bar{t} \leqslant \bar{\bar{t}} \leqslant d_P$. If P is such that $\rho_P \sigma \in \partial \mathscr{A}$, then $P \in \Pi^0$.

Axiom 4 excludes from this treatment mechanical processes involving restorations, neither phase-transitions of first order (e.g., melting or reforming), nor gluing or welding processes, that would modify the original nature of the body.

Theorem 1. $\forall P \in \Pi^0$ and $\sigma_0 \in \mathring{\mathscr{A}}$, $\forall a(\cdot, \cdot)$ for which $\inf\{a_{\sigma_0} \to \partial \mathscr{A}\}$ and $\inf\{a_{\sigma_2} \to \partial \mathscr{A}\}$ exist and are finite, being $\sigma_2 = \rho_{P^0P}\sigma_0$, $a(P^0P, \sigma_0)$ enjoys dissipative features with respect to process P^0P, i.e.,

$$a(P^0P, \sigma_0) > 0. \tag{17}$$

Proof. Because of the triangular inequality, for the definition of inf

$$\inf\{a_{\sigma_0} \to \partial \mathscr{A}\} \leqslant \inf\{a_{\sigma_0} \to \sigma_2\} + \inf\{a_{\sigma_2} \to \partial \mathscr{A}\}. \tag{18}$$

Now, for the definition of the damaging process $P \in \Pi^0$ and Axiom 4

$$\inf\{a_{\sigma_0} \to \sigma_2\} \geqslant \inf\{a_{\sigma_0} \to \partial \mathscr{A}\} - \inf\{a_{\sigma_2} \to \partial \mathscr{A}\} \geqslant 0. \tag{19}$$

The real number $a(P^0P, \sigma_0)$ is an element of the set $\{a_{\sigma_0} \to \sigma_2\}$. ■

Definition 2. Consider two states, σ and σ'; σ' is said to be a damaged state with respect to σ if σ and σ' can be connected by at least one path induced by a process which causes damage; i.e., if there exists $P \in \Pi^0$ such that $\rho_{P^0P}\sigma = \sigma'$.

Axiom 5. If σ' is a damaged state with respect to σ, σ' is attainable† from σ only by means of damaging processes such that $\rho_{P^0P}\sigma = \sigma'$.

† Because of the "physical" point of view adopted in this paper, only perfect approachability, i.e., attainability, is considered. Other possible weak forms of approachability can be considered in order to obtain greater generality.

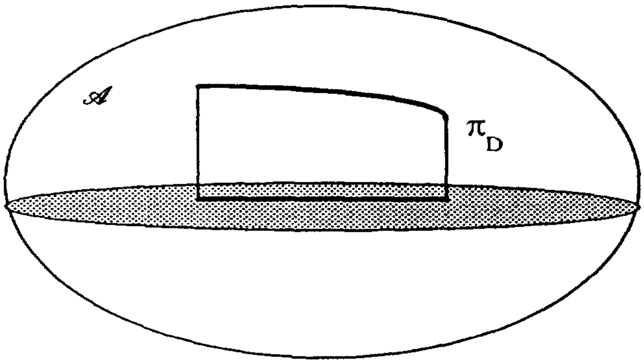

Fig. 2. Projection of the path onto the state variables subspace (schematic).

3.3. *Measures of damage*

The damaged state is described by a field or set of variables in both the internal variable and the microstructured models. Indicating with Σ' the subspace of Σ including all state variables (considered as fields) which describe the damaged state, a projection π_D can be considered, which selects the fields which describe the damage in the n-plet σ. It projects all paths in $\mathscr{A}$ on the subset of $\mathscr{A}$, indicated by $\mathscr{A}'$, which is in Σ' (see Fig. 2). If $\rho_P\sigma$ is a final state of a state transformation ρ_P:

$$\pi_D\rho_P\sigma \overset{\text{def}}{} \begin{cases} = \pi_D\sigma & \text{if } \exists P^{-1} \quad \text{such that } \rho_{P^{-1}}\rho_P\sigma = \sigma \\ \neq \pi_D\sigma & \text{in all the other cases.} \end{cases} \tag{20}$$

Nothing can be said about the projection of states which cannot be attained from each other at least in one sense, since their projection can coincide or differ. A criterion of distinction is given in the following where the symbol $[\{a_\sigma \to \mathscr{I}\} \mid \pi_D]$ indicates the set of all values of $a(\cdot,\cdot)$ calculated on the projection of the paths $\rho_{P'}\sigma$ (starting from the state σ and going into the states in $\mathscr{I}$, being $\mathscr{I}$ a subset of $\mathscr{A}$) on the space $\Sigma' \cap \mathscr{A} = \mathscr{A}'$.

Proposition 1. Let $P \in \Pi^0$ and $\sigma_1 = \rho_{P^0}{}_P\sigma$ then

$$\pi_D\sigma \neq \pi_D\sigma_1 \tag{21}$$

$$\inf[\{a'_\sigma \to \partial\mathscr{A}\} \mid \pi_D] > \inf[\{a'_{\rho_{P^0}{}_P\sigma} \to \partial\mathscr{A}\} \mid \pi_D] \tag{22}$$

$$\inf[\{a''_\sigma \to \partial\mathscr{A}\} \mid \pi_D] > \inf[\{a''_{\rho_{P^0}{}_P\sigma} \to \partial\mathscr{A}\} \mid \pi_D]. \tag{23}$$

In other words it is required that the amounts of energy and entropy necessary to bring the state $\rho_{P^0}{}_P\sigma$ (damaged with respect to σ) onto the ultimate region $\partial\mathscr{A}$ are smaller than those starting from σ.

Moreover, π_D must be such that

$$\text{sign}\inf\{a'_\sigma \to \rho_{P^0}{}_P\sigma\} = \text{sign}\inf[\{a'_\sigma \to \rho_{P^0}{}_P\sigma\} \mid \pi_D] \tag{24}$$

$$\text{sign}\inf\{a''_\sigma \to \rho_{P^0}{}_P\sigma\} = \text{sign}\inf[\{a''_\sigma \to \rho_{P^0}{}_P\sigma\} \mid \pi_D]. \tag{25}$$

The same level of damage is associated to two states σ_1 and σ_2 if

$$\inf[\{a'_{\sigma_1} \to \partial\mathscr{A}\} \mid \pi_D] = \inf[\{a'_{\sigma_2} \to \partial\mathscr{A}\} \mid \pi_D] \tag{26}$$

$$\inf[\{a''_{\sigma_1} \to \partial\mathscr{A}\} \mid \pi_D] = \inf[\{a''_{\sigma_2} \to \partial\mathscr{A}\} \mid \pi_D] \tag{27}$$

while σ_2 is a damaged state with respect to σ_1 if

$$\pi_D\sigma_1 \neq \pi_D\sigma_2 \tag{28}$$

$$\inf[\{a'_{\sigma_1} \to \partial\mathscr{A}\} \mid \pi_D] > \inf[\{a'_{\sigma_2} \to \partial\mathscr{A}\} \mid \pi_D] \tag{29}$$

$$\inf[\{a''_{\sigma_1} \to \partial\mathscr{A}\} \mid \pi_D] > \inf[\{a''_{\sigma_2} \to \partial\mathscr{A}\} \mid \pi_D]. \tag{30}$$

This statement holds even if they cannot be attained from each other at least in one sense.

In this way, the projector π_D establishes in $\mathscr{A}$ a partial order relation with respect to the damage. In this way, equal damage sets with respect to the energy and to the entropy can be recognized in the space $\mathscr{A}$. These sets are generalized damage level surfaces.

By using the projector π_D, it is possible to recognize that the composition between a cyclical process and a damaging process is a damaging process as shown by the following theorem.

Theorem 2. Let P be such that P^{-1} exists and $(P^{-1}P, \sigma) \in (\Pi \diamondsuit \Sigma)_{cycl}$, $\forall \sigma \in \mathscr{A}$. If $\bar{P} \in \Pi^0$, then

$$P_1 = P\bar{P} \in \Pi^0 \tag{31}$$

$$P_2 = \bar{P}P \in \Pi^0. \tag{32}$$

Proof. Let $P_1 = P\bar{P}$ be a process with P and $\bar{P}$ as defined above. Then, assuming $P^{-1} = P^0$†

$$\rho_{P_1^0 P_1}\sigma = \rho_{\bar{P}^0 P^{-1}}\rho_{P\bar{P}}\sigma = \rho_{\bar{P}^0 \rho_{P^{-1}P}}\rho_{\bar{P}}\sigma = \rho_{\bar{P}^0\bar{P}}\sigma. \tag{33}$$

If the action $a(\cdot,\cdot)$ satisfies the hypotheses of Theorem 1, the following inequality holds

$$\inf\{a_\sigma \to \partial\mathscr{A}\} > \inf\{a_{\rho_{\bar{P}^0\bar{P}\sigma}} \to \partial\mathscr{A}\} = \inf\{a_{\rho_{P_1^0 P_1 \sigma}} \to \partial\mathscr{A}\}. \tag{34}$$

As a consequence, $\sigma' = \rho_{P_1^{-1}P_1}\sigma$ is a damaged state with respect to σ. Thus, because of Axiom 5, $P_1 \in \Pi^0$.

Now, if $P_1 = \bar{P}P$, the following equations can be obtained

$$\rho_{P_2^0 P_2}\sigma = \rho_{P^{-1}}\rho_{\bar{P}^0\bar{P}}\rho_P\sigma \tag{35}$$

$$\pi_D\sigma = \pi_D\rho_P\sigma \neq \pi_D\rho_{\bar{P}^0\bar{P}}\rho_P\sigma = \pi_D\rho_{P^{-1}}\rho_{\bar{P}^0\bar{P}}\rho_P\sigma \tag{36}$$

and, because $\bar{P} \in \Pi^0$,

$$\inf[\{a_\sigma \to \partial\mathscr{A}\} \mid \pi_D] > \inf[\{a_{\rho_{P_2^0 P_2 \sigma}} \to \partial\mathscr{A}\} \mid \pi_D]. \tag{37}$$

As a consequence, $\rho_{P_2^0 P_2}\sigma$ is a damaged state with respect to σ and hence, for Axiom 5, $P_2 \in \Pi^0$.

The same result can be achieved also when $\rho_{\bar{P}^0\bar{P}}\rho_P\sigma \notin D(P^{-1})$. ■

Theorem 2 permits to introduce in Π some algebraic structures in the state space Σ and to affirm that states which are approachable one another by reversible (elastic, in particular) state transformations have the same possible damaged states. The latter statement is formally formulated in the following Corollary.

† If $P = P_1P_2$, then $P^{-1} = (P_1P_2)^{-1} = P_2^{-1}P_1^{-1}$.

Corollary. If $\mathrm{Inv}(\sigma)$ is the set defined as

$$\mathrm{Inv}(\sigma) \overset{\mathrm{def}}{=} \{\rho_P\sigma \mid P\in\Pi^0, \sigma\in D(P), \rho_P\sigma\in D(P^{-1}) \quad \text{and} \quad (P^{-1}P,\sigma)\in(\Pi\diamondsuit\Sigma)_{cycl}\} \tag{38}$$

i.e., the set of all states approachable from σ by a reversible process, then, $\forall\sigma'\in\mathrm{Inv}(\sigma)$,

$$\Pi^0\sigma' = \Pi^0\sigma \tag{39}$$

where

$$\Pi^0\sigma \overset{\mathrm{def}}{=} \{\rho_P\sigma \mid P\in\Pi^0, \sigma\in D(P)\}. \tag{40}$$

This Corollary is an obvious consequence of Theorem 2.

In this way every damaged state can be considered as an equivalent class of states with respect to reversible state transformations.

In most damage models in technical literature a damage potential is assumed to exist and is used in analogy with the theory of plasticity. In the present context, the following Theorem can be formulated on the above subject.

Theorem 3. The quantity

$$\Delta(\sigma_1,\sigma_2) \overset{\mathrm{def}}{=} |\inf\{a_{\sigma_1}\to\partial\mathscr{A}\}-\inf\{a_{\sigma_2}\to\partial\mathscr{A}\}| \tag{41}$$

is a semimetric in $\mathscr{A}$ and is equal to the variation of a weak lower potential† for all the actions $a(\cdot,\cdot)$ when they are calculated on the process P^0P, $P\in\Pi^0$, and fulfill the hypotheses of Theorem 1.

Proof. Axiom 3 and definition of $\Delta(\cdot,\cdot)$ yield

$$0\leqslant\Delta(\sigma_1,\sigma_2)=\Delta(\sigma_2,\sigma_1)<\infty \quad \forall\sigma_1 \quad \text{and} \quad \sigma_2\in\mathscr{A} \tag{42}$$

and $\Delta(\sigma_1,\sigma_1)=0$.

Moreover, putting $\inf_i = \inf\{a_{\sigma_i}\to\partial\mathscr{A}\}$, and assuming σ_1, σ_2, $\sigma_3\in\mathscr{A}$,

$$|\inf_1-\inf_3| = |\inf_1-\inf_2+\inf_2-\inf_3| \leqslant |\inf_1-\inf_2|+|\inf_2-\inf_3| \tag{43}$$

i.e.,

$$\Delta(\sigma_1,\sigma_3)\leqslant\Delta(\sigma_1,\sigma_2)+\Delta(\sigma_2,\sigma_3). \tag{44}$$

With regard to second part of theorem in concerned, since $\Delta(\cdot,\cdot)$ is defined on the whole $\mathscr{A}$ for Axiom 4, it suffices to note that, if $\sigma_2=\rho_{P^0P}\sigma_1$ and $P\in\Pi^0$,

$$\inf\{a_{\sigma_1}\to\partial\mathscr{A}\}\leqslant\inf\{a_{\sigma_1}\to\sigma_2\}+\inf\{a_{\sigma_2}\to\partial\mathscr{A}\} \tag{45}$$

then, because of Def. 1 and Th. 1,

$$\inf\{a_{\sigma_1}\to\sigma_2\}\geqslant|\inf\{a_{\sigma_1}\to\partial\mathscr{A}\}-\inf\{a_{\sigma_2}\to\partial\mathscr{A}\}|=\Delta(\sigma_1,\sigma_2) \tag{46}$$

† The word weak is used in order to take into account that the domain of the potential considered coincides with the whole $\mathscr{A}$ and not with Σ.

i.e.,

$$a(P^0 P, \sigma_1) + \varepsilon > \Delta(\sigma_1, \sigma_2) \quad \forall \varepsilon > 0. \tag{47}$$

Thus, in agreement with the definition of lower potential given in Section 2, the existence of a real function $Z(\cdot)$, defined in the whole $\mathscr{A}$ and such that

$$Z(\sigma_1) - Z(\sigma_2) = \Delta(\sigma_1, \sigma_2) \tag{48}$$

can be shown.

$Z(\cdot)$ is a lower potential for the action $a(P^0 P, \cdot)$ with respect to the space $\mathscr{A}$. ■

Another result strictly connected with the assumption of the current damage models can be obtained in the present context. Namely the existence of a real positive number Λ_σ that bounds the damage energy from below.

The quantity Λ_σ, defined by

$$0 \leqslant \Lambda_\sigma = \inf_{P^0 P, P \in \Pi^0} \{a'_\sigma \to \mathscr{A}\} < \infty \tag{49}$$

is associated to each state σ.

Λ_σ is the minimum energy necessary to damage the state σ according with Definition 1 and hence it can be called lower bound of damage energy.

Λ_σ and the set of all states which can be approachable starting from σ in a reversible way, $\mathrm{Inv}(\sigma)$, are related by the following theorem.

Theorem 5. $\forall \sigma_2 \in \mathrm{Inv}(\sigma_1)$

$$\inf \{a'_{\sigma_2} \to \sigma_1\} \leqslant \Lambda_{\sigma_2}. \tag{50}$$

Proof. Let σ_1, σ_2, σ_3 be state such that $\sigma_2 \in \mathrm{Inv}(\sigma_1)$ and $\sigma_3 \in \Pi^0 \sigma_1 \equiv \Pi^0 \sigma_2$, $\sigma_3 = \rho_{P_1^0 P_1} \sigma_1 = \rho_{P_2^0 P_2} \sigma_2$. For Axiom 4 and the properties of conservative actions the equality

$$\inf \{a_{\sigma_1} \to \sigma_2\} = -\inf \{a_{\sigma_2} \to \sigma_1\} \tag{51}$$

holds.

By substituting (50) into the triangular inequality

$$\inf \{a_{\sigma_1} \to \sigma_3\} \leqslant \inf \{a_{\sigma_1} \to \sigma_2\} + \inf \{a_{\sigma_2} \to \sigma_3\} \tag{52}$$

the following inequality

$$\inf \{a_{\sigma_1} \to \sigma_3\} \leqslant \inf \{a_{\sigma_2} \to \sigma_3\} - \inf \{a_{\sigma_2} \to \sigma_1\} \tag{53}$$

holds.

Because of $\inf \{a_{\sigma_1} \to \sigma_3\} \geqslant 0$ for Theorem 1,

$$\inf \{a_{\sigma_2} \to \sigma_3\} \geqslant \inf \{a_{\sigma_2} \to \sigma_1\}. \tag{54}$$

So, Theorem 5 is proved. ■

4. AN EXAMPLE: GENERALIZED PALMGREN–MINER'S RULE

A generalized form of the well-known Palmgren–Miner's empirical rule can be obtained in the present theoretical context.

On the basis of experimental results of fatigue tests on metal specimens, Miner (1945) developed a hypothesis by Palmgren and stated that a process composed by n cycles type

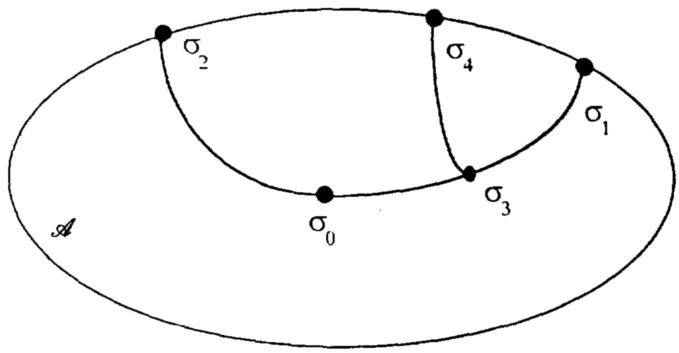

Fig. 3. Scenario of the Palmgren–Miner's rule.

A_1 and m cycles type A_2 lead to failure when

$$\frac{n}{N} + \frac{m}{M} = 1 \tag{55}$$

where N and M are the numbers of cycles respectively of type A_1 and A_2 that independently cause the failure of the specimen. This law holds, approximately, if the cycles are not correlated with each other and the dissipated work is the same in every cycle of the same type. These hypotheses are very strong from a theoretical point of view because the physical behavior is much more complex. However, the Palmgren–Miner's rule is currently used for design purposes in engineering practice because of its simplicity.

In the present Section, the Miner's rule is reformulated in the framework of the preceding axiomatic theory but shown to be slightly on the unsafe side. Because of the generality of the used formalism, this derivation holds for cycles of arbitrary type (not only stress or strain).

Consider two damaging processes P_1 and P_2 such that

(1) each process can be applied repeatedly, i.e.,

$$D(P_1) \bigcap R(P_1) \neq \varnothing \tag{56}$$

$$R(P_1) \bigcap D(P_2) \neq \varnothing \tag{57}$$

$$D(P_2) \bigcap R(P_2) \neq \varnothing \tag{58}$$

(2) the action value in each process is the same for any process of the same type, i.e.,

$$a(P_1^i, \rho_{P_1^{i-1}}\sigma) = a(P_1^{i-1}, \rho_{P_1^{i-2}}\sigma), \quad \forall i = 1 \ldots N \tag{59}$$

$$a(P_2^j, \rho_{P_1^{j-1}}\sigma) = a(P_2^{j-1}, \rho_{P_2^{j-2}}\sigma), \quad \forall j = 1 \ldots M \tag{60}$$

where N and M have been defined above.

Property 2 holds for at least one of the actions of Axiom 3.

(3) Another strong assumption is that the processes do not interfere with each other.

Now, let $P^n = PP \ldots P$, n times, and

$$\sigma_1 = \rho_{P_1^N}\sigma_0, \quad \sigma_2 = \rho_{P_2^M}\sigma_0, \quad \sigma_3 = \rho_{P_p^n}\sigma_0, \quad \bar{P} = PP_2^m P_1^n, \quad \text{and} \quad \sigma_4 = \rho_{\bar{P}}\sigma_0 \quad \text{(Fig. 3).}$$

Action additivity and assumption 2 above allow us to write

$$a(\bar{P},\sigma_0) = a(P_1^n,\sigma_0)+a(P_2^m,\sigma_0)+a(P,\rho_{P_2^mP_1^n}\sigma_0)$$
$$= na(P_1,\sigma_0)+ma(P_2,\sigma_0)+a(P,\rho_{P_2^mP_1^n}\sigma_0). \tag{61}$$

Then, if

$$a(\bar{P},\sigma_0) = a(P_2^M,\sigma_0) = Ma(P_2,\sigma_0) \tag{62}$$

$$a(\bar{P},\sigma_0) = a(P_1^N,\sigma_0) = Na(P_1,\sigma_0) \tag{63}$$

$$a(P_2,\rho_{P_1^M},\sigma_0) = a(P_2,\sigma_0) \tag{64}$$

the attainment of an unstable microcracking state $\sigma_4 \in \partial\mathscr{A}$ implies

$$\frac{n}{N}+\frac{m}{M} = 1-\eta \tag{65}$$

with

$$\eta = \frac{a(P,\rho_{P_2^mP_1^n}\sigma_0)}{a(\bar{P},\sigma_0)} \ll 1 \tag{66}$$

because m can be chosen such to minimize η.

Equation (65) is the generalized Palmgren–Miner's rule which is valid in the hypotheses 1, 2, 3: when $\eta = 0$, it coincides with the original eqn (55).

The result can be generalized considering instead of two processes P_1 and P_2 two sets of processes, $\{P_i'\}$, $i = 1\ldots N$; $\{P_j''\}$, $j = 1\ldots M$ such that

$$D(P_i')\bigcap R(P_{i-1}') \neq \varnothing \tag{67}$$

$$D(P_j'')\bigcap R(P_{j-1}'') \neq \varnothing \tag{68}$$

$$\bigcup_{i,j}[D(P_j'')\bigcap R(P_i') \neq \varnothing \tag{69}$$

and, put $\sigma_i = \rho_{P_i'\ldots P_1'}\sigma_0$ and $\sigma_j = \rho_{P_j''\ldots P_1''}\sigma_0$ such that

$$\inf\{a_{\sigma_i} \to \sigma_{i+1}\} = \inf\{a_{\sigma_{i-1}} \to \sigma_i\} \tag{70}$$

$$\inf\{a_{\sigma_j} \to \sigma_{j+1}\} = \inf\{a_{\sigma_{j-1}} \to \sigma_j\}. \tag{71}$$

This last property relaxes the classical assumption of equality of works in the previous treatment.

Now, let $E_\lambda(\sigma_0)$ be the set defined below

$$E_\lambda(\sigma_0) \stackrel{\text{def}}{=} \{\sigma \mid \inf\{a_{\sigma_0} \to \sigma\} = \lambda \in \mathscr{R}, 0 < \lambda < \infty\}. \tag{72}$$

With reference to Fig. 3 and putting $\sigma_1 = \rho_{P_N'\ldots P_1'}\sigma_0$, $\sigma_2 = \rho_{P_M''\ldots P_1''}\sigma_0$, $\sigma_3 = \rho_{P_n'\ldots P_1'}\sigma_0$, $\sigma_1 = \rho_P\rho_{P_{m+j}''\ldots P_1'}\sigma_0$, the following inequality holds

$$\inf\{a_{\sigma_0} \to \sigma_4\} \leqslant \inf\{a_{\sigma_0} \to \sigma_3\}+\inf\{a_{\sigma_3} \to \rho_{P_{m+j}''\ldots P_j''}\sigma_3\}+\inf\{a_{\rho_{P_{m+j}''\ldots P_j''}} \to \sigma_4\}. \tag{73}$$

If σ_1, σ_2, $\sigma_4 \in E_\lambda(\sigma_0)$, the properties of $\{P_i'\}$ and $\{P_j''\}$ can be used to show that the yielding of the state σ_4, according with the procedure described above, implies

$$\frac{n}{N}+\frac{m}{M}\geqslant 1-\eta' \tag{74}$$

with

$$0\leqslant \eta' = \frac{\inf\{a_{\rho_{P''_{m+j}\cdots P''_j\sigma_3}} \rightarrow \sigma_4\}}{\lambda} \ll 1. \tag{75}$$

5. CONCLUDING REMARKS

As recently state by Krajcinovic (1995), so far "little if anything has been done to formulate the axiomatic structure of Damage Mechanics". This paper intends to be a contribution in this direction: the proposed axiomatic construction provides a tentative framework to rationally establish some common fundamental properties of the models that have been developed in the existing literature on CDM. Although the theory presented is not yet exhaustive and requires further developments, the results illustrated in previous Sections have great generality and provide theoretical basis for many assumptions which are in damage models.

In a follow-up paper, Mariano (1996c) shows that this axiomatic framework is also useful to describe the behavior of elastic plastic materials. In particular it is emphasized that some of the foundations of the plasticity theory can be placed in a natural way into the present structure.

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